Int. J. Heat Mass Transfer. Vol. 13, pp. 717–723. Pergamon Press 1970. Printed in Great Britain

DISTRIBUTION OF CONCENTRATION IN FLOW THROUGH A CIRCULAR PIPE

D. H. CHUN

Mobil Research and Development Corporation, Field Research Laboratory, Dallas, Texas, U.S.A.

(*Received* 23 *July* 1969 *and in revised form* 29 *September* 1969)

Abstract—An analytical solution of concentration distribution in circular pipe flow is obtained for small Péclét numbers. A perturbation technique with a strained coordinate is applied. For Péclét numbers less than 0·1, the radial distribution of concentration is found to be practically uniform.

NOMENCLATURE

$\bar{C}$, concentration of the solute;
C_0, characteristic concentration;
C, dimensionless concentration, $\bar{C}/C_0$;
D, diffusion coefficient of the solute;
F, function defined by equation (6);
F_n, perturbation of F;
$\bar{F}$, Laplace transform of F;
$\bar{F}_n$, Laplace transform of F_n;
f_n, adjustable function, see equation (17);
g_1, g_2, g_3, shorthand notations, see equations (39)–(41);
J_0, zero order Bessel function of first kind;
J_1, first order Bessel function of first kind;
Pe, Péclét number, $U_{max} R/D$;
R, radius of tube;
$\bar{r}$, radial coordinate;
r, dimensionless radial coordinate, $\bar{r}/R$;
s, independent variable defined by equation (16);
$\bar{t}$, time;
t, dimensionless time, $\bar{t}D/R^2$;
U_{max}, maximum velocity in tube;
$\bar{x}$, longitudinal coordinate;
x, dimensionless longitudinal coordinate, $\bar{x}/R$;
ξ, strained coordinate defined by equation (17);
ξ_1, shorthand notation, see equation (36);
λ_n, eigenvalue, see equation (34).

INTRODUCTION

A soluble material introduced in a flow spreads out under the combined influence of the molecular diffusion due to Brownian motion and the convection due to the variation of bulk velocity. Dispersion of a solute in a circular tube was investigated first by Taylor [1–3] and later by several other investigators [8–10]. Recently, Gill *et al.* [11–17] extended this line of study considerably, using both theoretical and experimental methods.

Non-uniform concentration in a tube flow may disturb the otherwise parabolic velocity profile by including natural convection and further viscosity variation. A complete analysis of this problem requires consideration of momentum and mass transfer simultaneously. Gill *et al.* [13, 17] studied the effect of natural convection on dispersion and showed that its effect could be substantial for small Péclét number. However, in small capillaries of tight porous media, the Grashof number which is indicative of natural convection could also be small and perhaps small enough that the effect of natural convection could be neglected.

In this paper, we will simply neglect natural convection and use a perturbation technique for small Péclét number to find the local concentration distribution in a tube flow.

STATEMENT OF PROBLEM

Let us consider an axially symmetric mass transfer in a steady flow running through a circular pipe. The flow is initially free from the solute, and a solution of constant concentration starts flowing through one end of the pipe at time zero.

Interpreted in mathematical equations, the statement of the problem becomes

$$\frac{\partial C}{\partial t} + Pe(1 - r^2)\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}. \quad (1)$$

With initial and boundary conditions;

$$C = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad x > 0 \quad (2)$$

$$C = 0 \quad \text{at} \quad x \to \infty \quad (3)$$

$$C = 1 \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0 \quad (4)$$

$$\frac{\partial C}{\partial r} = 0 \quad \text{at} \quad r = 0, 1. \quad (5)$$

All the variables in the above equations are dimensionless and their notations are conventional. Refer to the nomenclature section for further clarification.

It is convenient to define a new dependent variable F such that

$$C = F(x, r, t)\exp\left[\frac{Pe}{4}\left(x - \frac{Pe}{4}t\right)\right]. \quad (6)$$

We then obtain a new set of differential equations;

$$\frac{\partial F}{\partial t} + Pe\left(\frac{1}{2} - r^2\right)\left(\frac{\partial F}{\partial x} + \frac{Pe}{4}F\right) = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial x^2} \quad (7)$$

$$F = 0 \quad \text{at} \quad t = 0 \quad (8)$$

$$F = 0 \quad \text{at} \quad x \to \infty \quad (9)$$

$$F = \exp\left(\frac{Pe^2}{16}t\right) \quad \text{at} \quad x = 0 \quad (10)$$

$$\frac{\partial F}{\partial r} = 0 \quad \text{at} \quad r = 0, 1. \quad (11)$$

Applying the Laplace transform, equations (7)–(11) yield

$$s\tilde{F} + Pe\left(\frac{1}{2} - r^2\right)\left(\frac{\partial \tilde{F}}{\partial x} + \frac{Pe}{4}\tilde{F}\right) = \frac{\partial^2 \tilde{F}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{F}}{\partial r} + \frac{\partial^2 \tilde{F}}{\partial x^2} \quad (12)$$

$$\tilde{F} = 0 \quad \text{at} \quad x \to \infty \quad (13)$$

$$\tilde{F} = \frac{1}{s - (Pe^2/16)} \quad \text{at} \quad x = 0 \quad (14)$$

$$\frac{\partial \tilde{F}}{\partial r} = 0 \quad \text{at} \quad r = 0, 1 \quad (15)$$

where $\tilde{F}$ is the transformed variable defined by

$$\tilde{F}(s) = \int_0^\infty F(t)\,e^{-st}\,dt. \quad (16)$$

At the present, an exact analytic solution of the above set of equations is difficult to find. Here, we will try to obtain an approximation for small Pe. For a set of differential equations which has a very small or very large parameter, it seems natural to try some of the perturbation techniques [4].

For our case, the solution of regular perturbation blows up for large x, and even singular perturbation (inner–outer expansion) does not seem to work. Instead, we will try to find a solution by straining the coordinate x.

SOLUTION

Now let us define a new strained coordinate

ξ and also perturb $\tilde{F}$ such that

$$\xi = x + \sum_{n=1}^{\infty} Pe^n f_n(x, r) \quad (17)$$

$$\tilde{F} = \tilde{F}_0 + \sum_{n=1}^{\infty} Pe^n \tilde{F}_n. \quad (18)$$

Functions f_n are unknown yet and quite arbitrary. Our strategy here is to determine f_n so as to make all perturbations $\tilde{F}_n$ except $\tilde{F}_0$ vanish. Then we will finally have

$$\tilde{F} = \tilde{F}_0. \quad (19)$$

First, we substitute equations (17) and (18) into equations (12)–(15), and follow the usual perturbation procedure.

The equations of zero order perperturbation become

$$s\tilde{F}_0 = \frac{\partial^2 \tilde{F}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{F}_0}{\partial r} + \frac{\partial^2 \tilde{F}_0}{\partial \xi^2} \quad (20)$$

$$\tilde{F}_0 = 0 \qquad \text{at} \quad \xi \to \infty \quad (21)$$

$$\tilde{F}_0 = \frac{1}{s - (Pe^2/16)} \qquad \text{at} \quad \xi = \sum_{n=1}^{\infty} Pe^n f_n(0, r) \quad (22)$$

$$\frac{\partial \tilde{F}_0}{\partial r} = 0 \qquad \text{at} \quad r = 0, 1. \quad (23)$$

To make the boundary condition (22) manageable, we put a restriction on f_n such that

$$f_n(0, r) = 0 \qquad n = 1, 2, \ldots \quad (24)$$

The zero order solution can be obtained by separation-of-variables; it is

$$\tilde{F}_0 = \frac{1}{s - \frac{Pe^2}{16}} \exp[-\sqrt{(s)}\xi] \quad (25)$$

The equations of the first order perturbations are

$$s\tilde{F}_1 - \left[\frac{\partial^2 \tilde{F}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{F}_1}{\partial r} + \frac{\partial^2 \tilde{F}_1^2}{\partial \xi^2}\right] = \left[-(\tfrac{1}{2} - r^2) + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{\partial^2 f_1}{\partial x^2}\right] \frac{\partial \tilde{F}_0}{\partial \xi} + 2\frac{\partial f_1}{\partial x}\frac{\partial^2 \tilde{F}_0}{\partial \xi^2} \quad (26)$$

$$\tilde{F}_1 = 0 \qquad \text{at} \quad \xi \to \infty \quad (27)$$

$$\tilde{F}_1 = 0 \qquad \text{at} \quad \xi = 0 \quad (28)$$

$$\frac{\partial \tilde{F}_1}{\partial r} = -\frac{\partial \tilde{F}_0}{\partial \xi}\frac{\partial f_1}{\partial r} \qquad \text{at} \quad r = 0, 1. \quad (29)$$

Besides restriction (24), we may put more restrictions on f_1 to make $\tilde{F}_1$ vanish. We set

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - 2\sqrt{(s)}\frac{\partial f_1}{\partial x} + \frac{\partial^2 f_1}{\partial r^2} = \frac{1}{2} - r^2 \quad (30)$$

$$\frac{\partial f_1}{\partial r} = 0 \qquad \text{at} \quad r = 0, 1. \quad (31)$$

Restrictions (30) and (31) force the right hand sides of equation (26) and boundary condition (29) to become zero. Therefore, the first order solution becomes trivial:

$$\tilde{F}_1 = 0. \quad (32)$$

From equations (24), (30) and (31), we can find f_1 by separation-of-variables:

$$f_1 = \sum_{n=1}^{\infty} \frac{4J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \{1 - \exp[(\sqrt{(s)} - \sqrt{(s + \lambda_n^2)})x]\} \quad (33)$$

where λ_n is the root of

$$J_1(\lambda_n) = 0 \quad (34)$$

in ascending order excluding zero.

In principle, we may carry on the same procedures to obtain higher f_n. However, differential equations of f_n for $n \geqslant 2$ become highly complex, and we will stop here.

In sum, equations (17), (19), (25) and (33) yield

$$\tilde{F} = \frac{1}{s - \frac{Pe^2}{16}} \exp\{-\sqrt{(s)}[x + Pe\xi_1 + \ldots]\} \quad (35)$$

where

$$\xi_1 = \frac{1}{16}\left\{\frac{1}{3} - (1 - r^2)^2\right\} - \sum_{n=1}^{\infty} \frac{4J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \exp\{[\sqrt{(s)} - \sqrt{(s + \lambda_n^2)}]\}. \quad (36)$$

In obtaining expression (36), the following identity was used:

$$\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} = \frac{1}{64}\left\{\frac{1}{3} - (1 - r^2)^2\right\}.$$

The inverse Laplace transform of $\tilde{F}$ gives

$$F = \frac{1}{2\pi i} \int_{r - i\infty}^{r + i\infty} \tilde{F}\, e^{st}\, ds. \tag{37}$$

$\tilde{F}$ has a pole at $s = Pe^2/16$ and branch points at $s = 0$, λ_n^2 $(n = 1, 2, \ldots)$. Applying Cauchy's integral formula [5–7] to equation (37), equation (6) yields

$$C = \exp\left[-\frac{Pe^2}{4} g_1\right] - \frac{2}{\pi} \exp\left[\frac{Pe}{4}\left(x - \frac{Pe}{4} t\right)\right] \sum_{m=1}^{\infty} \int_{\lambda_{m-1}}^{\lambda_m} \frac{\rho}{\rho^2 + (Pe^2/16)}$$
$$\exp\left[-\rho(\rho t + Pe\, g_2)\right] \sin \rho\, g_3\, d\rho \tag{38}$$

where

$$g_1 = \frac{1}{16}\left\{\frac{1}{3} - (1 - r^2)^2\right\}$$
$$\sum_{n=1}^{\infty} \frac{4 J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \exp\left\{\frac{Pe}{4} - \sqrt{\left[\left(\lambda_n^2 + \frac{Pe^2}{16}\right)\right]}\right\} x \tag{39}$$

$$g_2 = \sum_{n=1}^{m-1} \frac{4 J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \sin\left(\rho - \sqrt{[\rho^2 - \lambda_n^2]}\right) x$$
$$+ \sum_{n=m}^{\infty} \frac{4 J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \exp\left[-x\sqrt{(\lambda_n^2 - \rho^2)}\right] \sin x\rho \tag{40}$$

$$g_3 = x + Pe\left\{\frac{1}{16}\left[\frac{1}{3} - (1 - r^2)^2\right]\right\}$$
$$- \sum_{n=1}^{m-1} \frac{4 J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \cos\left[\rho - \sqrt{(\rho^2 - \lambda_n^2)}\right] x$$
$$- \sum_{n=m}^{\infty} \frac{4 J_0(\lambda_n r)}{\lambda_n^4 J_0(\lambda_n)} \exp\left[-x\sqrt{(\lambda_n^2 - \rho^2)}\right] \cos x\rho\ . \tag{41}$$

RESULTS

Equation (38) was evaluated with the aid of a CDC 6400 computer, and the results were plotted on Figs. 1–4. All the variables and parameters on the plots are dimensionless.

On Fig. 1, isoconcentration lines are shown with Péclét number, $Pe = 0{\cdot}1$, at time $t = 0{\cdot}2$. We see that the radial variation of concentration is negligible.

Equation (38) was obtained by a perturbation applicable to small Péclét numbers. However, to exaggerate the effect of convection in dispersion, Figs. 2 and 3 were plotted for relatively high Péclét numbers. The deviation from the true solution corresponding to high Péclét numbers was not checked, but the approximate solution shows proper aspects of dispersion. The radial variation of concentration increases as Péclét numbers increase, that is, as the convection effect becomes strong. Also the longitudinal dispersion increases with increasing Péclét numbers.

In Fig. 4, the concentration at the center of the tube was plotted along the longitudinal axis with time and Péclét numbers as parameters. The lines with $Pe = 0$ represent the pure diffusional cases. At the beginning the differences of concentration due to differences of Péclét numbers are small, but increase with time.

A similar problem was investigated by Gill *et al.*, applying the finite difference technique. The range of Péclét number in their investigation is higher than ours and their results were presented in a somewhat different fashion. However, our radial distribution of concentration seems consistent with their results.

ACKNOWLEDGEMENT

The author wishes to thank Mobil Research and Development Corporation for permission to publish this work.

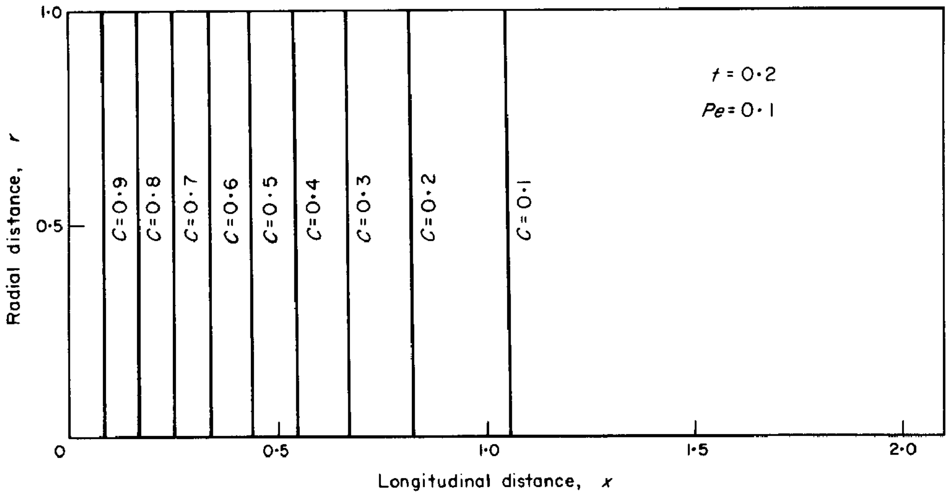

FIG. 1. Isoconcentration distribution at $t = 0{\cdot}2$ and $Pe = 0{\cdot}1$.

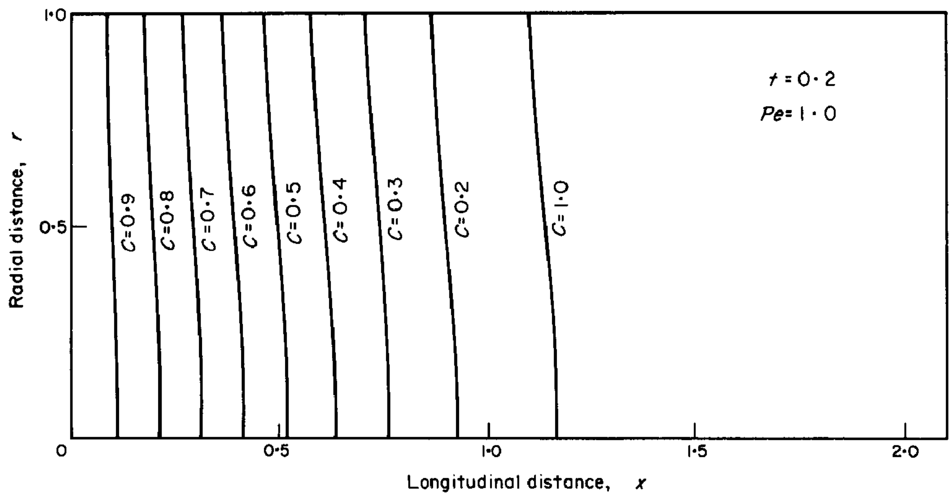

FIG. 2. Isoconcentration distribution at $t = 0{\cdot}2$ and $Pe = 1{\cdot}0$.

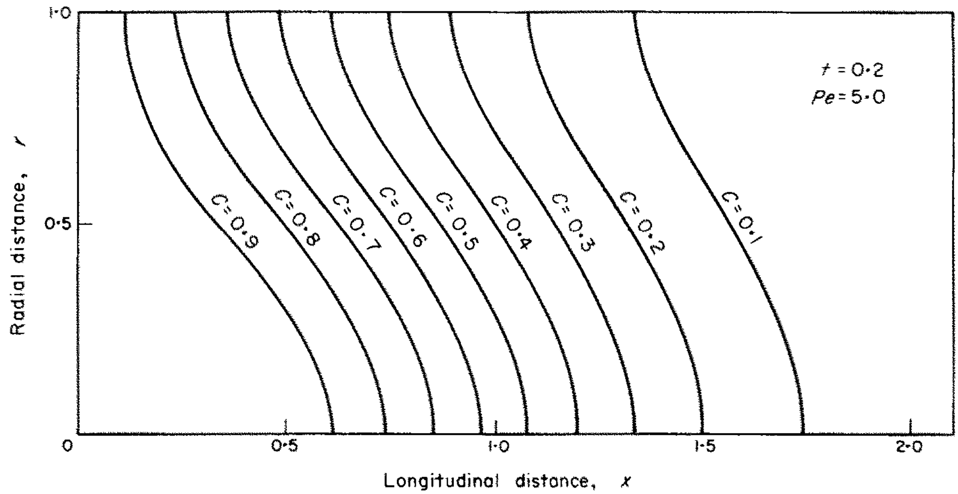

FIG. 3. Isoconcentration distribution at $t = 0{\cdot}2$ and $Pe = 5{\cdot}0$.

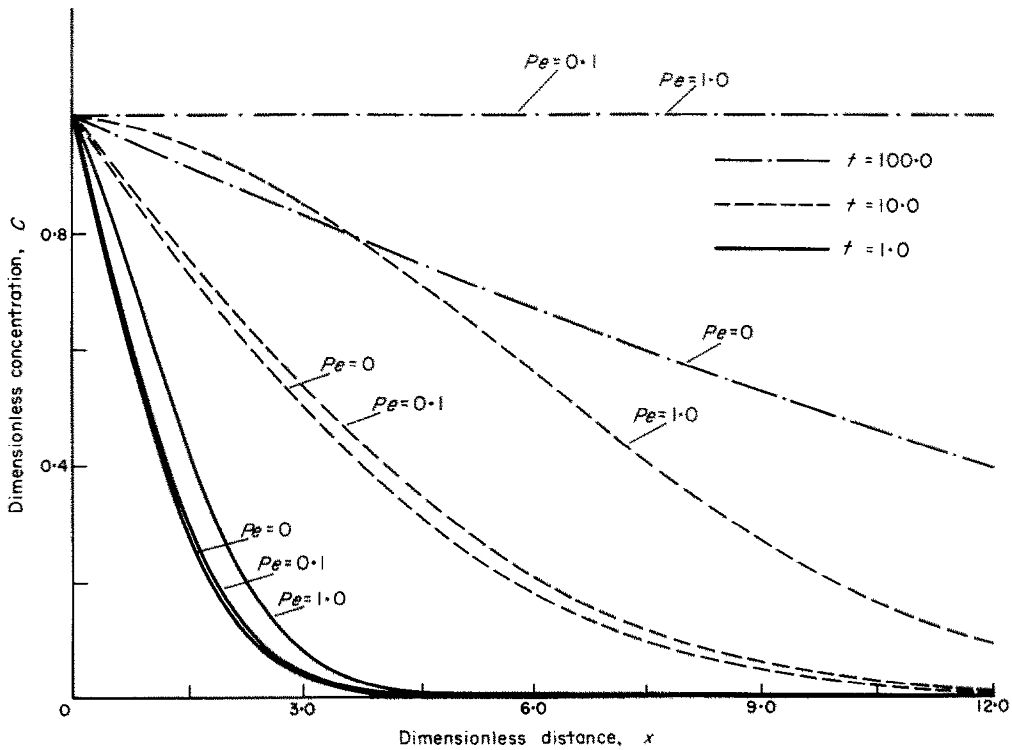

FIG. 4. Longitudinal concentration distribution at the tube center with time and Péclét numbers as parameters.

REFERENCES

1. G. Taylor, Dispersion of soluble matter in solvent flowing slowly through a tube, *Proc. R. Soc.* A **219**, 186–203 (1953).
2. G. Taylor, The dispersion of matter in turbulent flow through a pipe, *Proc. R. Soc.* A **223**, 446–468 (1954).
3. G. Taylor, Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion, *Proc. R. Soc.* A **225**, 473–477 (1954).
4. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*. Academic Press, New York (1964).
5. R. V. Churchill, *Complex Variables and Applications*. McGraw-Hill, New York (1960).
6. N. W. McLachlan, *Complex Variable Theory and Transform Calculus*. Cambridge (1955).
7. H. S. Carslaw and J. C. Jaeger, *Operational Methods and Applied Mathematics*. Oxford (1941).
8. R. Aris, On the dispersion of a solute in a fluid flowing through a tube, *Proc. R. Soc.* A **235**, 67–97 (1956).
9. G. F. Carrier, On diffusive convection in tubes, *Q. Appl. Math.* **14**, 108–112 (1956).
10. M. J. Lighthill, Initial development of diffusion in Poiseuille flow, *J. Inst. Maths. Applics* **2**, 97–108 (1966).
11. V. Ananthakrishnan, W. N. Gill and A. J. Barduhn, Laminar dispersion in capillaries, *A.I.Ch.E. Jl* **11**, 1063–1072 (1965).
12. W. N. Gill, Laminar dispersion in capillaries: Part II, *A.I.Ch.E. Jl* **12**, 906–915 (1966).
13. N. S. Reejhsinghani, W. N. Gill and A. J. Barduhn, Part III. Experiments in horizontal tubes including observations on natural convection effects, *A.I.Ch.E. Jl* **12**, 916–921 (1966).
14. W. N. Gill, A note on the solution of transient dispersion problems, *Proc. R. Soc.* A **298**, 335–339 (1967).
15. W. N. Gill, Analysis of axial dispersion with time variable flow, *Chem. Engng Sci.* **22**, 1013–1017 (1967).
16. W. N. Gill and V. Ananthakrishnan, Laminar dispersion in capillaries: Part IV., *A.I.Ch.E. Jl* **13**, 801–807 (1967).
17. N. S. Reejhsinghani, A. J. Barduhn and W. N. Gill, Laminar dispersion in capillaries: part V, *A.I.Ch.E. Jl* **14**, 100–109 (1968).

DISTRIBUTION DE CONCENTRATION DANS L'ÉCOULEMENT À TRAVERS UN TUYAU CIRCULAIRE

Résumé—Une solution analytique de la distribution de concentration dans un écoulement à travers un tuyau circulaire est obtenu pour de petits nombres de Péclet. On applique une technique de perturbation avec une coordonnée déformée. Pour des nombres de Péclet plus petits que 0, 1, on trouve que la distribution radiale de la concentration est pratiquement uniforme.

KONZENTRATIONSVERTEILUNG IN EINER STRÖMUNG DURCH ROHRE VON KREISQUERSCHNITT

Zusammenfassung—Eine analytische Lösung für die Konzentrationsverteilung bei Strömungen in Rohren von Kreisquerschnitt liess sich für kleine Péclét-Zahlen erhalten. Es wurde eine Störungsmethode mit gestreckter Koordinate angewandt. Für Péclét-Zahlen kleiner als 0,1 ergab sich die radiale Konzentrationsverteilung praktisch als gleichförmig.

РАСПРЕДЕЛЕНИЕ КОНЦЕНТРАЦИИ ПРИ ТЕЧЕНИИ В КРУГЛОЙ ТРУЬЕ

Аннотация—Получено аналитическое решение распределения концентрации при течении в круглой трубе. Для малых значений числа Пекле использован метод теории возмущений. Найдено, что для чисел Пекле, меньших О, I, радиальное распределение концентрации практически однородно.